EM II: DC Circuits and Capacitors

FIZIKA SJPO Training

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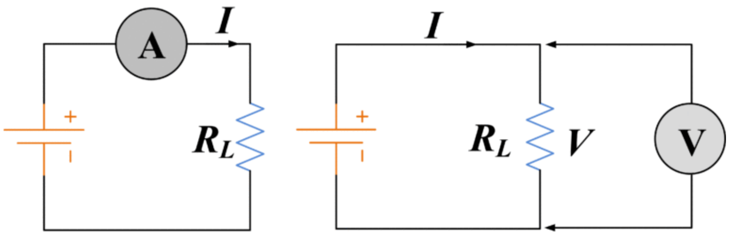
1 Notes

1.1 DC Circuits

The basic quantities in circuit analysis at I (current), V (voltage) and R (resistance). **Current** is measured by an **ammeter connected in series,** while **voltage** is measured by a **voltmeter connected in parallel.**



You can think of a series connection as circuit elements connected "side by side" and a parallel connection as circuit elements "stacked on top of each other".

1.1.1 Resistivity and Conductivity

Sometimes, it may also be useful to define ρ (resistivity) and σ (conductivity), which follow:

$$R = \frac{\rho L}{A} \tag{1}$$

$$\sigma = \frac{1}{\rho} \tag{2}$$

where L is the length of the material and A is the cross-sectional area of the material.

ρ and σ are convenient, because they are **material properties** (i.e. only dependent on the material).

1.1.2 Ohm's Law

Ohm's Law directly links the three basic quantities:

$$V = IR \tag{3}$$

In some books, you may instead see Ohm's Law written as

$$\mathbf{J} = \sigma \mathbf{E} \tag{4}$$

Here, $\mathbf{E}$ is just the electric field, while $\mathbf{J}$ is defined as the **current density,** which can just be thought of as current per area.

Clearly, Equation (3) is more useful in circuit analysis, while Equation (4) is more useful when you are finding fields and potentials.

More rigorously, I can be defined as the rate of flow of charge:

$$I = \frac{\Delta q}{\Delta t} \tag{5}$$

Remark. Zero current **does not** imply zero voltage, and zero voltage **does not** imply zero current! (Think of examples to verify this.)

Example 1.1 (SJPO 2018)**.** A battery of emf 1.80 V has an internal resistance of 0.75 Ω. When a wire with circular cross-section of 1.5 cm in diameter and 125 m long is connected across the terminals of this battery, the current is 0.25 A. What is the resistivity of the material from which the wire is made?

Solution. We simply apply both Equations (1) and (3) in tandem. The effective (total) resistance is given by $R_{\text{eff}} = R + r$, where R is the resistance of the wire and r is the internal resistance of the battery. Hence,

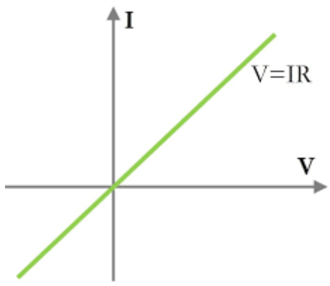
$$I = \frac{V}{R_{\text{eff}}} = \frac{V}{R+r} = \frac{V}{\frac{\rho L}{A} + r} \implies \rho = \frac{A}{L}\left(\frac{V}{I} - r\right) = \frac{\pi r^2}{L}\left(\frac{V}{I} - r\right)$$

$$= \frac{\pi\left(\frac{1.5\times10^{-2}}{2}\right)^2}{125}\left(\frac{1.80}{0.25} - 0.75\right) = 9.1 \times 10^{-6}\,\Omega\,\text{m}$$

1.1.3 I-V Characteristics

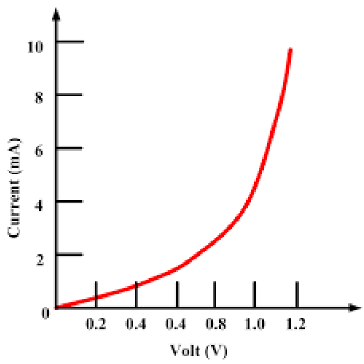
When we analyse circuit elements, we are usually interested in varying I and V. An **I-V characteristic** is a graph of I against V.

For an **ohmic, linear resistor,** the I-V characteristic is a linear graph passing through the origin, since R is constant in Equation (3):



The resistance is simply just the *inverse* of the gradient of the graph.

For other **non-linear components,** their I-V characteristics may be **curves.**



In this case, the resistance is not constant and can be calculated at every point by evaluating $\frac{V}{I}$ **at that point.** For the graph above, resistance is decreasing as the gradient is increasing.

Remark. You may be tempted to write Ohm's Law in terms of infinitesimals:

$$R = \frac{\mathrm{d}V}{\mathrm{d}I} = \text{gradient of the graph of } V \text{ against } I$$

However, this is **wrong** by definition! Ohm's Law gives the relationship between the current and voltage at an instant, and has nothing to do with $\frac{dV}{dI}$ (which is meaningless).

1.1.4 Electromotive Force (Emf)

Interestingly, **emf is NOT a force,** even though its name has "force" in it!

The emf, ε, is the work done by a source to drive a unit charge around a circuit:

$$\varepsilon = \frac{W}{q} \tag{6}$$

Remark. Emf is often confused with voltage. While both have units of volts (V), emf is a characteristic of the **source,** while voltage is measured across any two points in a circuit!

We will revisit emf when we discuss electromagnetic induction. For now, it suffices to think of it as the voltage across a source.

1.1.5 Power Dissipated by Circuit Elements

For any circuit element, the electric power dissipated is

$$P = VI \tag{7}$$

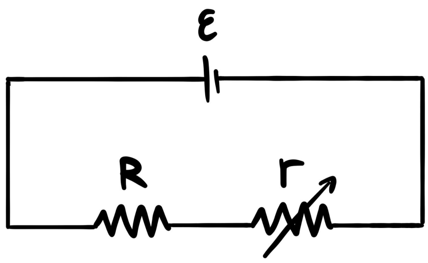
Only for ohmic resistors, we may write

$$P = \frac{V^2}{R} \tag{8}$$

$$P = I^2 R \tag{9}$$

Another thing to note is that the **total power** dissipated in a circuit (this includes power dissipated by sources, which is usually negative) is 0. This must be true by energy conservation.

Now, consider the following circuit with a battery of emf ε and two resistors R and r connected in series. Suppose R is fixed, and r can be freely adjusted.



It turns out that maximum power transfer to the resistor r occurs when

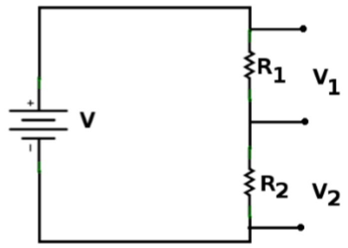
$$r_{\text{max power}} = R \tag{10}$$

Equation (10) is simply the **maximum power transfer theorem.** The power transferred to a load resistance is maximum when it is equal to the source resistance in series with it.

1.1.6 Voltage and Current Divider Rules

We shall begin our discussion on circuit analysis. In circuit analysis, our objective is to solve for currents in branches of circuits, or the potential differences between circuit elements.

Consider the following circuit:



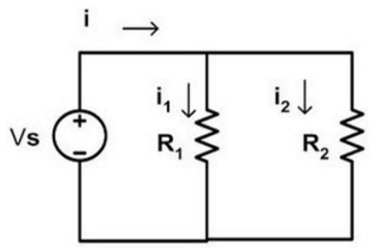
The **voltage divider rule** states that

$$V_1 = \frac{R_1}{R_1 + R_2} V, \; V_2 = \frac{R_2}{R_1 + R_2} V \tag{11}$$

or, in general,

$$V_i = \frac{R_i}{\sum R} V \tag{12}$$

Consider another circuit:



The **current divider rule** states that

$$I_1 = \frac{R_2}{R_1 + R_2} I, \; I_2 = \frac{R_1}{R_1 + R_2} I \tag{13}$$

or, in general,

$$I_i = \frac{R_i}{\sum \frac{1}{R}} I \tag{14}$$

Remark. In practice, you don't need these rules, and you can just work everything out with Ohm's Law. But hopefully these rules help to simplify and speed up your calculations.

1.1.7 Kirchhoff's Laws

Kirchhoff's Laws are the fundamentals to circuit analysis. There are two laws - **Kirchhoff's Current Law (KCL)** and **Kirchhoff's Voltage Law (KVL).**

KCL: The sum of currents entering/leaving a node (a point in the circuit) is 0. Mathematically,

$$\sum I = 0 \tag{15}$$

The assignment of signs (for entering/leaving) is arbitrary, as long as you are consistent. In this handout, we shall assign outgoing currents as positive and incoming currents as negative.

KVL: The sum of potential **drops** across a loop is 0. Mathematically,

$$\sum \Delta V = 0 \tag{16}$$

The assignment of signs is positive for potential drops (positive to negative) and negative for potential gains (negative to positive).

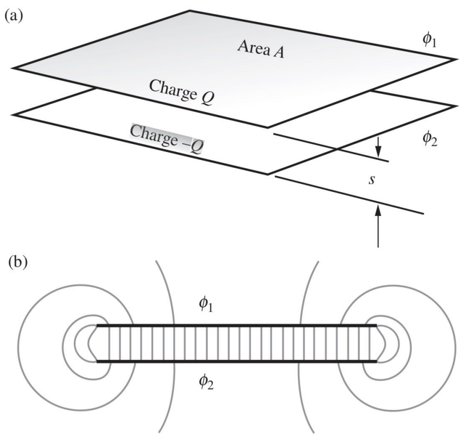
1.2 Capacitors

A capacitor is a device that stores electrical energy using two electrically charged conducting plates separated by an insulator. This creates an electrostatic field within the capacitor that stores the electrical energy.

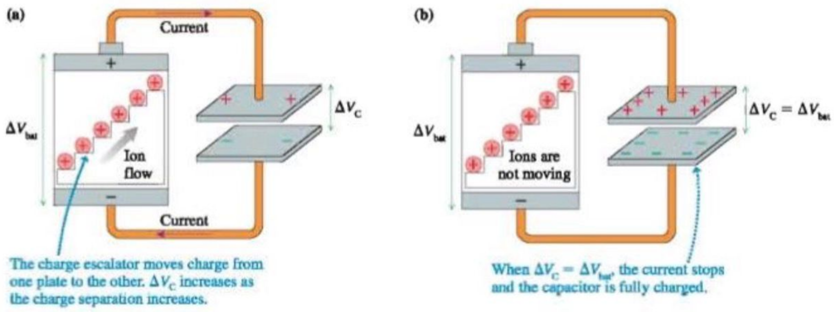
Suppose we have a capacitor consisting of two oppositely charged plates $\pm Q$, such that the potential difference between the two plates is ΔV. We refer to the capacitor having charge Q and voltage ΔV. We then define the **capacitance** of the capacitor as

$$C = \frac{Q}{\Delta V} \tag{17}$$

whose SI units are farads (F = C V^{-1}). It turns out that capacitance is purely *geometric* property, i.e. only depends on the geometry of the two terminals.

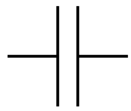


A capacitor is able to be charged and discharged, in a similar fashion to a battery. We can charge a capacitor by connecting its plates to opposite terminals of a battery, as shown. The battery is able to act as a "charge escalator" that does work by moving charge from one plate to the other, charging the capacitor. As positive charge builds up on the positive terminal of the capacitor, the repulsive force on incoming positive charges increases until eventually no new charge is able to enter and the capacitor is fully charged.



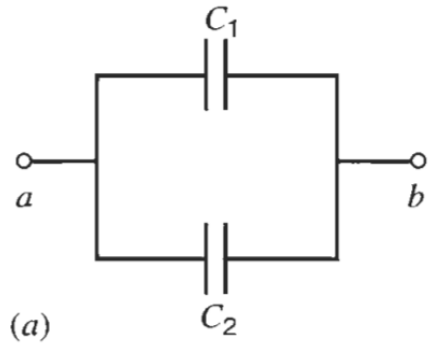
Once the capacitor is fully charged, since the charges in the wire are no longer moving and in equilibrium, the plates of the capacitor are at the same potential as their respective connected terminals, so the voltage ΔV of the capacitor is equal to the voltage ε of the battery. Then, when the capacitor is disconnected from the battery, the charge and voltage of the capacitor is maintained.

The circuit diagram of a capacitor looks as follows:



1.2.1 Capacitor Combinations

Capacitors may be connected to other capacitors in series or parallel. Similar to resistors, we can derive expressions for the **effective capacitance** of capacitor combinations.



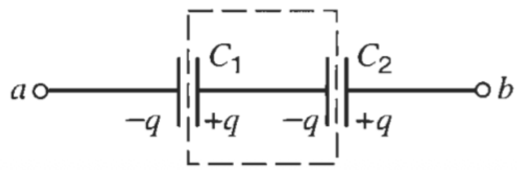
Let us first consider several capacitors in **parallel**, with capacitances $C_1, C_2, \ldots, C_n$. Then, the capacitors share the same potential difference ΔV. Since all the positive plates of the capacitors are connected (and vice versa), the effective capacitor of the capacitor combination has the same potential difference ΔV and total charge Q that is the sum of the charges on each capacitor $Q_1, Q_2, \ldots, Q_n$. Hence, by definition,

$$Q = Q_1 + Q_2 + \cdots + Q_n = C_1\Delta V + C_2\Delta V + \cdots + C_n\Delta V = (C_1 + C_2 + \cdots + C_n)\,\Delta V$$

so the effective capacitance of the capacitors in parallel is

$$C_{\text{eff}} = C_1 + C_2 + \cdots + C_n \tag{18}$$

Note: Do realise that this is slightly different from resistors: the effective resistance adds across resistors in *series*, not parallel.



Now, let us consider several capacitors in **series**, with capacitances $C_1, C_2, \ldots, C_n$. Since the wires connecting the capacitors are neutral (as we would typically assume), the positive plate of a capacitor must have opposite charge to the negative plate of an adjacent capacitor (and vice versa), implying that all capacitors have the same charge Q. Then, for the effective capacitor of the capacitor combination, the outermost plates also have charge Q, i.e. the effective capacitor has charge Q, and the potential difference ΔV of the effective capacitor must be the sum of the potential differences of the capacitors $\Delta V_1, \Delta V_2, \ldots, \Delta V_n$. Hence, by definition,

$$\Delta V = \Delta V_1 + \Delta V_2 + \cdots + \Delta V_n = \frac{Q}{C_1} + \frac{Q}{C_2} + \cdots + \frac{Q}{C_n} = Q\left(\frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_n}\right)$$

so the effective capacitance of the capacitors in series is

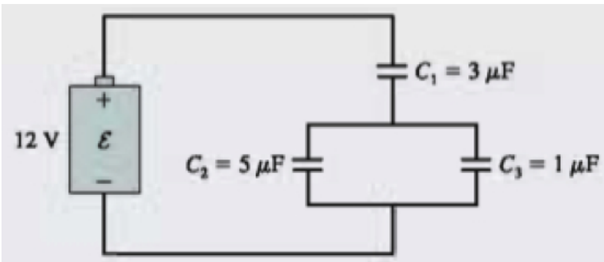
$$\frac{1}{C_{\text{eff}}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_n} \tag{19}$$

Note: Do realise that this is (again) slightly different from resistors: the reciprocal of the effective resistance adds across resistors in *parallel*, not series.

We can use this information to determine the charges and potential differences of capacitors in any arrangements, by determining the effective capacitances of capacitors in series or parallel, and noting that:

- capacitors in parallel (and their effective capacitor) share the same potential difference, and
- capacitors in series (and their effective capacitor) share the same charge.

Example 1.2. In the circuit consisting of three capacitors and a battery as shown below, determine (a) the potential difference ΔV_1 across capacitor C_1, (b) the charge Q_2 on capacitor C_2.



Solution. (a) The two capacitors in parallel can be replaced with an effective capacitor of capacitance

$$C_{23} = C_2 + C_3 = 6\ \mu\text{F}$$

Then, the new capacitor will be in series with capacitor C_1, so the three capacitors can be replaced with a single capacitor of effective capacitance

$$C = \left(\frac{1}{C_1} + \frac{1}{C_{23}}\right)^{-1} = 2\ \mu\text{F}$$

Then this effective capacitor is directly connected to the battery, so its charge is

$$Q = C\Delta V = C\varepsilon = 24\ \mu\text{C}$$

This is the same charge as capacitor C_1, so its charge is also

$$Q_1 = Q = 24\ \mu\text{C}$$

i.e. the potential difference across capacitor C_1 is

$$\Delta V_1 = \frac{Q_1}{C_1} = 8 \text{ V}.$$

(b) The charge on Q_1 is also the same as the total charge on C_2 and C_3, so

$$Q_{23} = Q = 24 \ \mu\text{C}$$

which gives

$$\Delta V_{23} = \frac{Q_{23}}{C_{23}} = 4 \text{ V}$$

This is the same potential difference as capacitor C_2, so

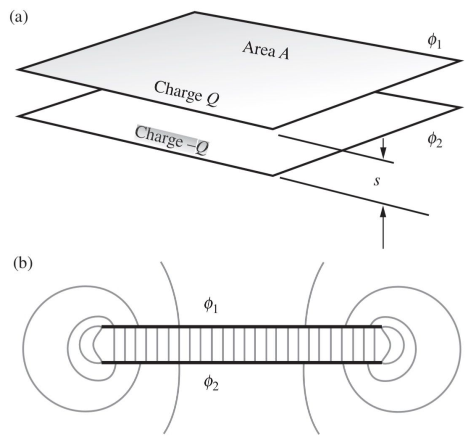
$$\Delta V_2 = \Delta V_{23} = 4 \text{ V}$$

and so its charge is

$$Q_2 = C_2 \Delta V_2 = 20 \ \mu\text{C}.$$

1.2.2 Parallel Plate Capacitors

Consider two identical thin conducting plates of opposite charges $\pm Q$ arranged parallel to each other, each with area A and separated by distance d. Let us also suppose that the distance between the plates is small enough such that the electric field between the plates can be approximated as uniform, i.e. $d \ll \sqrt{A}$.



Recall that the magnitude of the electric field due to a large, thin and uniformly charged plate is $E_{\text{plate}} = \sigma/2\varepsilon_0$ where σ is the charge density of the plate. (This result is derived using Gauss's law.) Since there are two oppositely charged conducting plates, the electric field in the region between the plates points from the positive to the negative plate and has magnitude

$$E = 2E_{\text{plate}} = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\varepsilon_0 A}$$

by superposition. Furthermore, since the electric field is uniform, we have

$$E = \frac{\Delta V}{d}$$

by definition. Therefore,

$$\frac{\Delta V}{d} = \frac{Q}{\varepsilon_0 A} \quad \Longrightarrow \quad Q = \frac{\varepsilon_0 A}{d} \Delta V$$

Comparing this with the definition of capacitance, we have the capacitance of the parallel-plate capacitor is

$$C = \frac{\varepsilon_0 A}{d} \tag{20}$$

where ε_0 is the permittivity of free space.

This example should meaningfully illustrate the process of calculating the capacitance for a given configuration of two oppositely charged conductors.

Example 1.3. Determine the capacitance of a capacitor consisting of two concentric spherical conducting shells, where the inner radius of the outer shell is a and the outer radius of the inner shell is b.

Solution. Suppose the inner shell has charge Q and the outer shell has charge $-Q$. If we consider the outer shell in isolation, since there is no charge in the cavity of the shell, by Gauss's law and spherical symmetry, there is no electric field in the cavity, i.e. the potential is the same everywhere in the cavity. Hence, the potential difference between the two shells is unaffected by the charge $-Q$ on the outer shell, and we can just consider the potential due to the inner shell.

The potential due to the inner shell of charge Q at radius r is

$$V(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$$

by Gauss's law. Therefore, the potential difference between the two shells, which have radii b and a respectively, is

$$\Delta V = V(b) - V(a) = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{b} - \frac{1}{a}\right)$$

and so the capacitance is

$$C = \frac{Q}{\Delta V} = \frac{4\pi\varepsilon_0}{1/b - 1/a} = \frac{4\pi\varepsilon_0 ab}{a-b}. \tag{21}$$

Note that even though we assumed the capacitor to have some unknown charge Q, the capacitance we obtained does not depend on Q. This is as expected since capacitance is a geometric property of the capacitor.

1.2.3 Energy

The important of capacitors comes from the fact that they are able to store **energy**. By considering the work required to bring some amount of charge from the negative plate to the positive plate across the potential difference of the capacitor, we can show that the energy U stored in a capacitor of capacitance C, being equal to the total work required to bring the capacitor from zero charge to charge Q, is

$$U = \frac{Q^2}{2C} = \frac{1}{2}C(\Delta V)^2 \tag{22}$$

This energy is being stored in the electric field between the oppositely charged plates of the capacitor. For example, for a parallel-plate capacitor of area A and distance d, since the electric field between the plates is $E = \frac{\Delta V}{d}$, the energy stored is

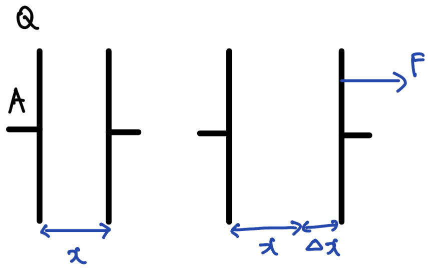
$$U = \frac{1}{2}\frac{\varepsilon_0 A}{d}(Ed)^2 = \frac{1}{2}\varepsilon_0 E^2 Ad \tag{23}$$

Since the volume of the region between the parallel plates is $V = Ad$, we can conclude that the *energy per unit volume* (or *energy density*) of the electric field within the capacitor is

$$u = \frac{1}{2}\varepsilon_0 E^2 \tag{24}$$

It turns out that energy is stored not just in the electric field within a capacitor but in fact within all electric fields (in a vacuum), with amount per volume given by this expression.

Example 1.4. A parallel-plate capacitor of area A and distance x is charged to a charge of Q with a battery. The capacitor is then disconnected from the battery, the negative plate is fixed in position and the positive plate is pulled away by an external agent at constant speed such that the distance between the plates increases by a small amount Δx. (a) Determine the increase in energy stored by the capacitor. (b) By equating the increase in energy stored to the work done by the agent, determine the force on the positive plate by the negative plate.



Solution. (a) Since the capacitor is disconnected from the battery, the charges $\pm Q$ on its plates remains the same, so the increase in energy is simply

$$\Delta U = U_{\text{f}} - U_{\text{i}} = \frac{Q^2}{2}\left(\frac{1}{C_{\text{f}}} - \frac{1}{C_{\text{i}}}\right) = \frac{Q^2}{2}\left(\frac{x+\Delta x}{\varepsilon_0 A} - \frac{x}{\varepsilon_0 A}\right) = \frac{Q^2 \Delta x}{2\varepsilon_0 A}.$$

(b) The work done by the external agent must go into the increase in energy stored in the capacitor. Since the external agent pulls the capacitor plate at constant speed, it exerts a force equal to that between the plates. Over a small distance, the force F does not change by a significant amount, so the work done by the agent is simply $W = F\Delta x$. Equating this value to the increase in energy,

$$W = \Delta U \quad \Longrightarrow \quad F\Delta x = \frac{Q^2 \Delta x}{2\varepsilon_0 A} \quad \Longrightarrow \quad F = \frac{Q^2}{2\varepsilon_0 A}.$$

Now, the following example illustrates an important note about charging capacitors and energy.

Example 1.5. Consider a capacitor of capacitance C being charged by a battery of voltage ε. (a) Determine the work done by the battery in charging the capacitor. (b) Determine the energy stored in the capacitor. (c) Are your answers equal? If so, explain why they should be. If not, briefly account for the discrepancy.

Solution. (a) The battery transports charge

$$Q = C\Delta V = C\varepsilon$$

across its terminals to charge the capacitor. By definition, the work done by the battery is

$$W = Q\varepsilon = C\varepsilon^2.$$

(b) The energy stored in the capacitor is simply

$$U = \frac{1}{2}C(\Delta V)^2 = \frac{1}{2}C\varepsilon^2.$$

(c) The work done by the battery and the energy stored in the capacitor are in fact different! This may initially seem to contradict the conservation of energy, but this can be resolved by considering the non-ideality of the wires in the circuit.

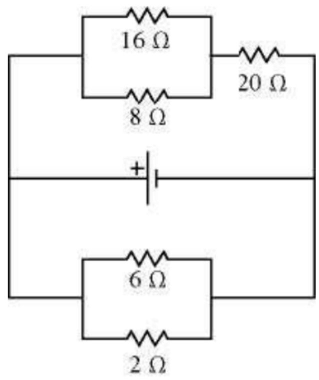
With perfectly ideal and indestructible wires, a circuit only consisting of a battery and a capacitor will tend to "oscillate", as the capacitor will actually charge past voltage ε and begin to discharge. However, with non-ideal wires that have a small resistance, the wires emit power that dissipate energy to the surroundings. This allows the capacitor-battery system to eventually reach equilibrium, where the capacitor remains at constant voltage ε. Here, we can then infer that the total energy the wires dissipate in the circuit is

$$U_{\text{dissipated}} = W - U = \frac{1}{2}C\varepsilon^2$$

which accounts for the discrepancy between the two values.

2 Problems

Problem 2.1. (SJPO 2010) The current of the circuit in the 8-Ω resistor is 0.5 A. What is the current in the 2-Ω resistor?



(A) 9.5 A

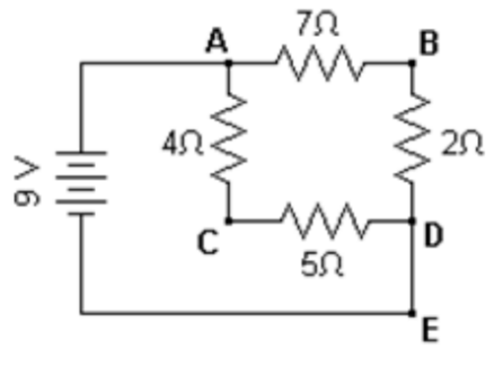
(B) 0.75 A

(C) 2.25 A

(D) 4.5 A

(E) 6.4 A

Problem 2.2. (SJPO 2011) A 9-volt battery is connected to four resistors to form a simple circuit as shown below. What would be the electric potential at point C with respect to point B in the circuit?



(A) +7 V

(B) +3 V

(C) 0 V

(D) −3 V

(E) −7 V

Problem 2.3. (SJPO 2012) A battery of emf 1.50 V has an internal resistance of 1.20 Ω. When a wire with circular cross-section of 2 mm in diameter and 250 m long is connected across the terminals of this battery, the current in the circuit is 0.4 A. What is the resistivity of the material from which the wire is made?

(A) 2.40×10^{-8} Ωm

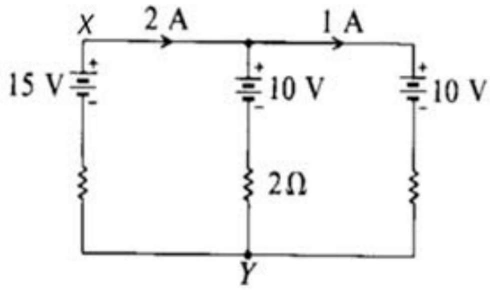
(B) 2.80×10^{-8} Ωm

(C) 3.20×10^{-8} Ωm

(D) 3.60×10^{-8} Ωm

(E) 4.00×10^{-8} Ωm

Problem 2.4. (SJPO 2012) In the circuit shown below, the emfs of the batteries are given, as well as the currents in the outside branches and the resistance in the middle branch. What is the magnitude of the potential difference between X and Y?



(A) 16 V

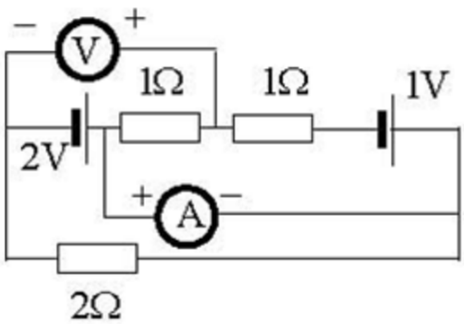
(B) 12 V

(C) 15 V

(D) 8 V

(E) 4 V

Problem 2.5. (SJPO 2013) The readings from the voltmeter and ammeter from the circuit shown below are



(A) 2.0 V, 1.0 A

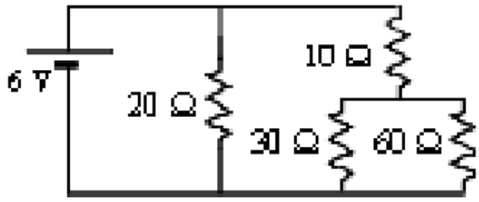
(B) 1.0 V, 0.5 A

(C) 1.5 V, 0.5 A

(D) 1.5 V, 1.0 A

(E) 1.0 V, 2.0 A

Problem 2.6. (SJPO 2015) Four resistors and a 6-V battery are arranged as shown in the circuit diagram. Through which resistor(s) does the smallest current in the circuit pass through?



(A) 10 Ω resistor

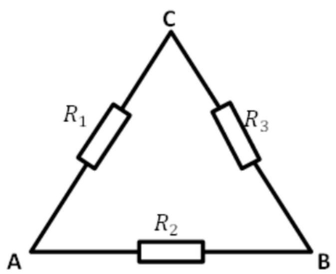
(B) 20 Ω resistor

(C) 30 Ω resistor

(D) 60 Ω resistor

(E) Same for both 30 Ω and 60 Ω resistors

Problem 2.7. (SJPO 2015) 3 resistors of the same resistance $R_1 = R_2 = R_3 = R$ forms a triangular circuit as shown. When one of the resistances is changed, measurements show that $R_{AB} = R_{AC}$, $R_{BC} = R/2$. Which is the resistance that has been changed and what is its new resistance?



(A) $R_1 = R/3$

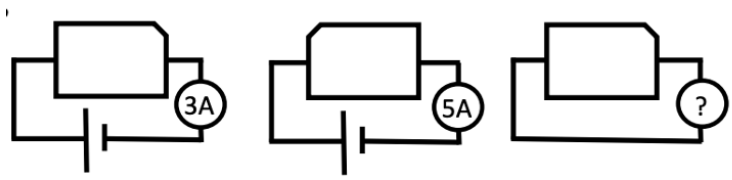
(B) $R_2 = R/3$

(C) $R_2 = 2R/3$

(D) $R_3 = R/3$

(E) $R_3 = 2R/3$

Problem 2.8. (SJPO 2016) A black box consists of a battery with EMF less than 12 volts and a resistor connected in series with the two ends of the circuit sticking out of the black box. If the ends of the black box are connected to a power supply of 12V, the current flowing through is 5A. If the connection to the black box is reversed, the current flowing through is 3A. What will be the current flowing through if the ends of the black box are connected by a wire?



(A) 1.0A

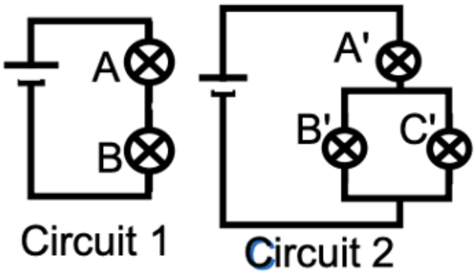
(B) 1.5A

(C) 2.0A

(D) 2.4A

(E) 4.0A

Problem 2.9. (SJPO 2016) The circuits below contain identical bulbs and batteries. It is known that the bulbs resistance increase with temperature. Compare the brightness of bulbs A and B in circuit 1 to the brightness of bulb A' and B' in circuit 2.



(A) $A > A',\ B > B'$

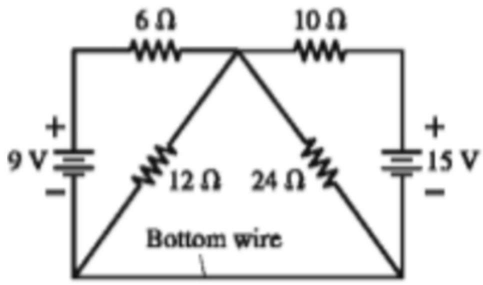
(B) $A > A',\ B < B'$

(C) $A = A',\ B < B'$

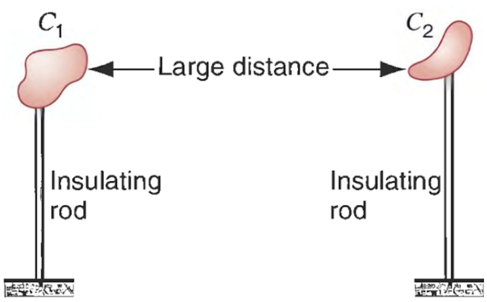
(D) $A < A',\ B = B'$

(E) $A < A',\ B > B'$

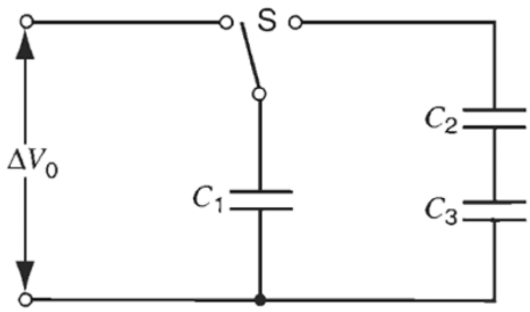
Problem 2.10. How much current flows through the bottom wire in the figure below, and in which direction?



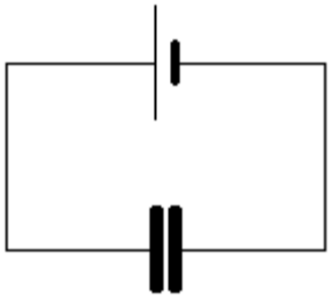
Problem 2.11. (HRK 30-Q7) You have two isolated conductors, each of which has a certain capacitance; see figure below. If you join these conductors by a fine wire, how do you calculate the capacitance of the combination? In joining them with the wire, have you connected them in series or parallel?



Problem 2.12. (HRK 30-P6) When switch S is thrown to the left in the figure below, the plates of the capacitor C_1 acquire a potential difference ΔV_0. C_2 and C_3 are initially uncharged. The switch is now thrown to the right. What are the final charges q_1, q_2, q_3 on the corresponding capacitors?

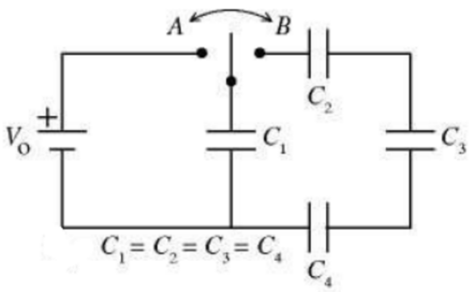


Problem 2.13. (SJPO 2009) A parallel-plate capacitor is connected to a battery as shown. What will happen if the separation of the plates is increased?



	Capacitance	Voltage	Charge
(A)	decreases	decreases	decreases
(B)	decreases	unchanged	decreases
(C)	decreases	decreases	increases
(D)	increases	unchanged	decreases
(E)	increases	unchanged	increases

Problem 2.14. (SJPO 2011) The four identical capacitors in the circuit shown below are initially uncharged. $V_{1,2,3,4}$ are the potential differences across $C_{1,2,3,4}$ and $Q_{1,2,3,4}$ are the final charges stored in $C_{1,2,3,4}$ respectively. The switch is then thrown first to position A, and then to position B. After this is done:



(A) $V_1 = V_0$

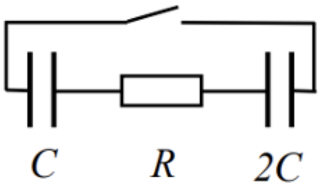
(B) $V_1 > V_2 > V_3 > V_4$

(C) $V_1 + V_2 + V_3 = V_4 = V_0$

(D) $Q_1 = Q_2 = Q_3 = Q_4$

(E) $Q_1 = 3Q_3$

Problem 2.15. (SJPO 2017) A capacitance C is initially charged to voltage V (i.e. the potential difference across its terminals is V) with charge Q. The energy stored in the capacitor is initially E. It is connected via a resistance R to another capacitor with capacitance $2C$. What is the final energy stored in the two capacitors after a long time t?



(A) $E/3$

(B) $E/2$

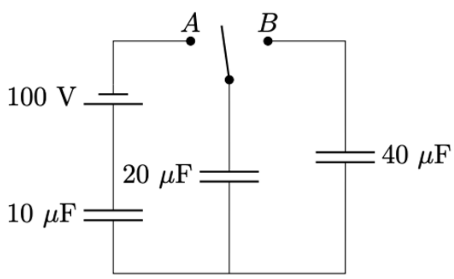
(C) E

(D) $E - \frac{1}{2}\frac{V^2}{R}t$

(E) $E - \frac{1}{8}\frac{V^2}{R}t$

Problem 2.16. Four identical metal plates are located in air at equal distances d from one another. The area of each plate is equal to A. Find the capacitance of the system between points A and B if the plates are interconnected as shown.

Problem 2.17. (SJPO 2024) Three initially uncharged capacitors are connected with a battery as shown in the circuit below. The switch is first set to position A, and after a long time, the switch is then set to position B. Find the final charge on the 40 μF capacitor.



(A) 220 μC

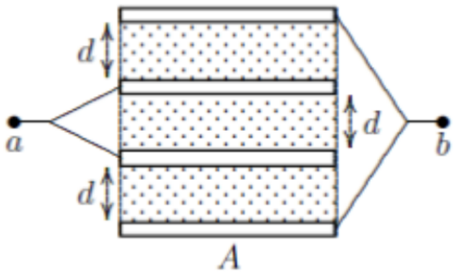
(B) 330 μC

(C) 440 μC

(D) 530 μC

(E) 670 μC

Problem 2.18. (SJPO 2015) Four large conducting plates, each of area A, are placed an equal distance d apart as shown in the following figure. Given that the permittivity of the medium filling the space between the plates is ϵ, what is the capacitance of this arrangement, assuming that the two terminals a and b of the capacitor are connected to the plates as shown?



(A) $\dfrac{\epsilon A}{2d}$

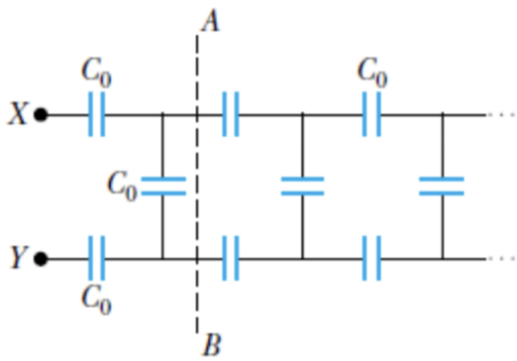
(B) $\dfrac{\epsilon A}{d}$

(C) $\dfrac{2\epsilon A}{d}$

(D) $\dfrac{3\epsilon A}{d}$

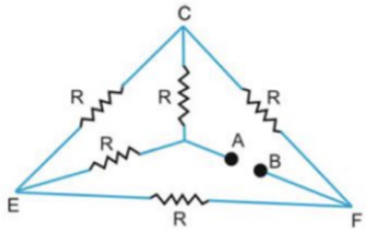
(E) $\dfrac{4\epsilon A}{d}$

Problem 2.19. Determine the effective capacitance between terminal X and Y of the infinite series of capacitors shown in the figure below. Each capacitor has capacitance C_o. Hint: imagine that the infinite ladder is cut at line AB, and think about what the effective capacitance of the section to the right of AB will be.

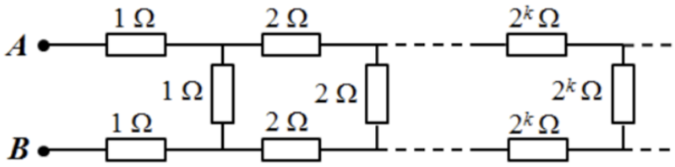


Problem 2.20. The problems from here on out are challenging.

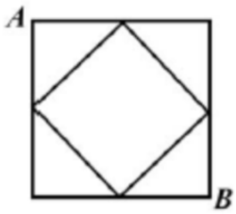
(a) Find the effective resistance between A and B.



(b) Find the resistance between points A and B of an infinite circuit shown. The resistance of the resistors in each loop is twice those of the previous loop on its left.



(c) A wire has linear resistance ρ (in Ω/m). Find the resistance between points A and B if the length of each side of the big square is d meters.



(d) A battery has an emf of 6 V. The battery is connected in series with an ammeter and a voltmeter. If a certain resistor is connected in parallel with the voltmeter, the voltmeter reading decreases by a factor 3, and the ammeter reading increases by the same factor. What is the initial reading V of the voltmeter? All elements of the circuit have unknown internal resistances.

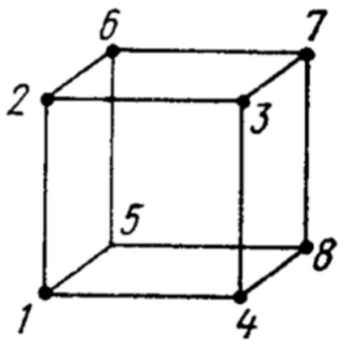
Problem 2.21. Find the resistance of a wire frame shaped as a cube as shown below when measured between points:

(a) 1 and 7,

(b) 1 and 2, and

(c) 1 and 3.

The resistance of each edge of the frame is R.



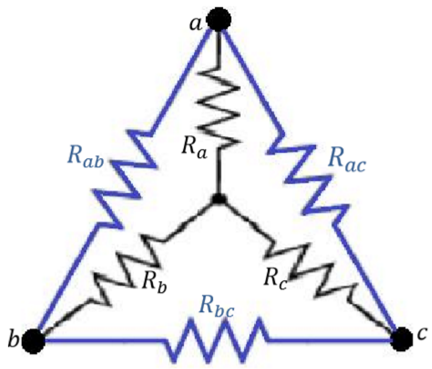
3 Appendix

3.1 Tricks For Calculating R_{eff}

Here, we detail many tricks that commonly appear in Olympiad questions for calculating R_{eff} in a wide variety of situations.

3.1.1 Δ-Y and Y-Δ Transformations

Consider three resistors, either in a Δ-shaped arrangement or a Y-shaped arrangement below:



The **Δ-Y and Y-Δ transformations** relate the resistances in the Δ configuration (in blue) and the Y configuration (in black):

$$R_a = \frac{R_{ab}R_{ac}}{R_{ab}+R_{bc}+R_{ac}}, \quad R_b = \frac{R_{ab}R_{bc}}{R_{ab}+R_{bc}+R_{ac}}, \quad R_c = \frac{R_{ac}R_{bc}}{R_{ab}+R_{bc}+R_{ac}} \tag{25}$$

$$R_{ab} = \frac{R_aR_b+R_bR_c+R_aR_c}{R_c}, \quad R_{bc} = \frac{R_aR_b+R_bR_c+R_aR_c}{R_a}, \quad R_{ac} = \frac{R_aR_b+R_bR_c+R_aR_c}{R_b} \tag{26}$$

In the special case where all three resistances are equal, we have

$$R_\Delta = 3R_\text{Y} \tag{27}$$

The proof of these transformations don't require anything beyond knowing resistors in series and parallel. (Prove them yourself as an exercise!)

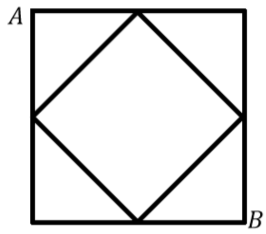
Remark. Often, the Δ-Y transformation is more useful, since resistor networks involving Δs are hard to deal with directly.

3.1.2 Equipotential Points and Symmetry

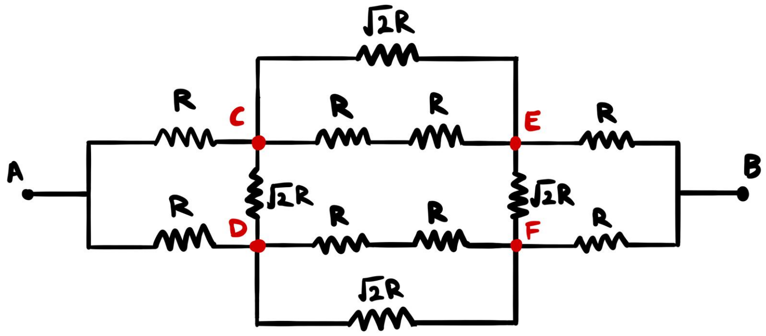
When simplifying complex circuits, spotting **symmetry** is very important. It allows you to identify **equipotential points** (points at the same potential), which can simplify whole branches of circuits!

The example below illustrates how to spot and utilise equipotential points.

Example 3.1 (SJPO Special Round 2009)**.** The shape shown in the figure is made of wire of constant cross section. The side of the bigger square is a m, and a 1 m length of wire has a resistance ρ. Determine the resistance between points A and B.

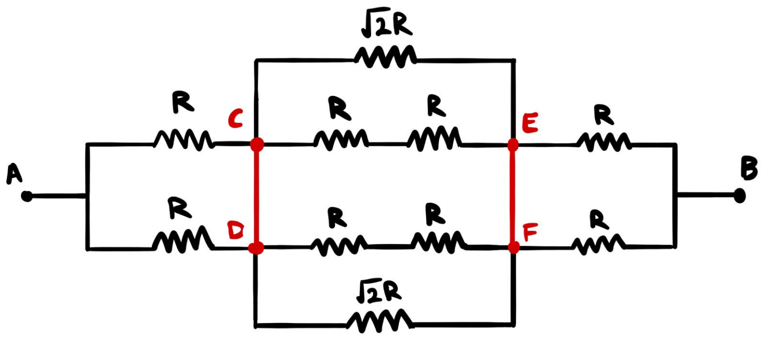


Solution. We can first simplify this by drawing out the circuit diagram. Let $R = \frac{\rho a}{2}$:



Notice that there is a high level of symmetry here. The points C and D are symmetric, and so are the points E and F. As such, there shouldn't be a preference for current to flow in either direction, so **no current flows** between these two pairs of points! (We call C, D and E, F pairs of equipotential points.)

As such, we can completely ignore the $\sqrt{2}R$ resistances between these pairs of points, and effectively join them with a wire (so that they are at the same potential):



This circuit is easy to evaluate as it is just series/parallel combinations. You should obtain:

$$R_{AB} = \frac{R}{2} + \frac{1}{2\left(\frac{1}{\sqrt{2}R}\right) + 2\left(\frac{1}{2R}\right)} + \frac{R}{2} = \sqrt{2}R = \frac{\sqrt{2}\rho a}{2}$$

3.1.3 Current Injection and Superposition

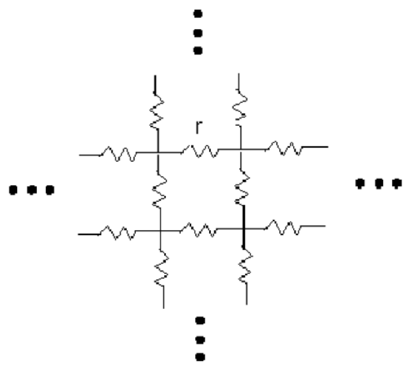
When dealing with even more complicated networks, it may be useful to consider two separate scenarios:

1. Put current I into one node.
2. Extract current I from another node.

We can analyse these scenarios separately to find the potential differences we are interested in. Afterwards, using superposition, we can "add" these scenarios together. Finally, using $V = IR_{\text{eff}}$ and equating the V found from superposition, we can determine R_{eff}.

This may sound a little confusing, but the example below should illustrate what it means.

Example 3.2 (SPhO 2003)**.** Consider the infinite square array of resistors each of resistance r as shown in the figure. Find the equivalent resistance between two neighbouring points separated by a resistor r.



Solution. Let's focus our attention on the points left and right to the resistor labelled r.

Suppose a current I flows in via the left point. By symmetry, a current $\frac{I}{4}$ flows through each of the resistors connected to that point. This contributes a potential difference of

$$V_{\text{left}} = \frac{Ir}{4}$$

between the two points.

Now, suppose we extract a current I via the right point. By symmetry, a current $\frac{I}{4}$ flows through each of the resistors connected to that point. This contributes a potential difference of

$$V_{\text{right}} = \frac{Ir}{4}$$

between the same two points.

We can then superpose the two cases together. By doing so, we simply add up the potential differences:

$$V = V_{\text{left}} + V_{\text{right}} = \frac{Ir}{2}$$

At the same time, $V = IR_{\text{eff}}$. Hence,

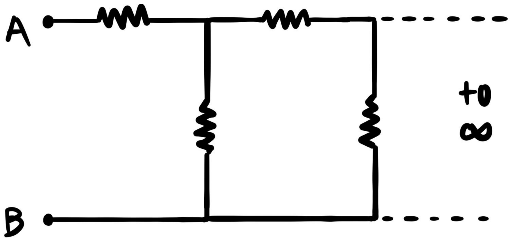
$$IR_{\text{eff}} = \frac{Ir}{2} \quad \Longrightarrow \quad R_{\text{eff}} = \frac{r}{2}$$

Remark. The last step is true, because by superposing the two cases together, we are essentially viewing the circuit as a constant current source of I connected between the two points.

3.1.4 Infinite Networks

Another common type of Olympiad question is the **infinite network.** To solve such problems, you want to use the concept of **self-similarity** (i.e. the circuit contains itself).

Example 3.3. Determine the effective resistance between the two terminals of the infinite resistor ladder as shown below. Each resistor has a resistance R.



Solution. Notice that since the ladder repeats to infinity, you can replace a part of the ladder with itself. Suppose the effective resistance is R_{eff}. Then, R_{eff} is the same as having R in series with a resistance of R in parallel with R_{eff}.

In particular, the following equation is satisfied:

$$R_{\text{eff}} = R + \frac{RR_{\text{eff}}}{R + R_{\text{eff}}} \quad \Longrightarrow \quad R_{\text{eff}}^2 - RR_{\text{eff}} - R^2 = 0$$

We can solve this quadratic equation, and take the (physically meaningful) positive root to get:

$$R_{\text{eff}} = \left(\frac{1 + \sqrt{5}}{2} \right) R$$

3.1.5 Negative Resistance

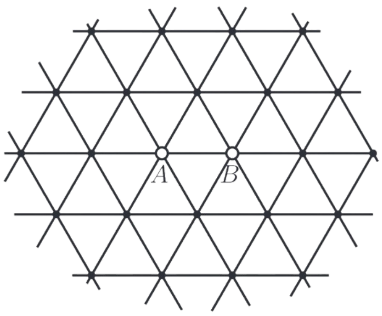
The last concept is negative resistance. This is especially useful in a situation that has resistors cut off, but would have otherwise been symmetric.

Essentially, you want to treat "missing resistances" as negative. There are two possibilities:

1. **Connect R and $-R$ in series:** This will give 0 resistance (a short circuit).
2. **Connect R and $-R$ in parallel:** This will give ∞ resistance (an open circuit).

The example below illustrates.

Example 3.4 (Kalda). (i) Determine the effective resistance between two neighbouring nodes in an infinite triangular lattice, as shown in the figure below. The resistance of the wire connecting any two neighbouring nodes is R. (ii) Now, suppose the wire connecting nodes A and B is cut. Determine the new effective resistance.



Solution. (i) We shall solve the first part in the spirit of Example 1.8.

Suppose a current I flows in via A. By symmetry, a current $\frac{I}{6}$ flows through each of the wires connected to A. This contributes a potential difference of

$$V_A = \frac{IR}{6}$$

between points A and B.

Now, suppose we extract a current I via B. By symmetry, a current $\frac{I}{6}$ flows through each of the wires connected to B. This contributes a potential difference of

$$V_B = \frac{IR}{6}$$

between points A and B.

Superposing the two cases together,

$$V = V_A + V_B = \frac{IR}{3}$$

At the same time, $V = IR_{\text{eff}}$. Hence,

$$IR_{\text{eff}} = \frac{IR}{3} \quad \Longrightarrow \quad R_{\text{eff}} = \frac{R}{3}$$

(ii) Notice that all the symmetry is broken by the cut wire. This inspires us to think about connecting a negative resistance.

Imagine putting two resistors of R and $-R$ in parallel between A and B, in place of the cut wire. This has the same effect as cutting the wire, since the effective resistance is ∞.

The circuit now becomes $\frac{R}{3}$ and $-R$ in parallel. Thus,

$$R_{\text{eff}} = \frac{\frac{1}{3}(-1)}{\frac{1}{3} - 1} R = \frac{R}{2}$$